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# ON CRITICAL PATH WITH FUZZY WEIGHTS

***Abstract.*** *This work deals with the critical path analysis in a fuzzy environment. The durations of the activities in a project are considered to be partially linearized Gaussian fuzzy numbers. A ranking method based on a certain crisp value called valuation is taken into account. It is used together with the Bellman-Kalaba algorithm to determine the minimum time required to complete all stages of a given project. The theoretical problems approached in the paper are followed by a numerical application.*

***Keywords*****:** *fuzzy number, ranking fuzzy numbers, Gaussian fuzzy number, valuation, critical path method, Bellman-Kalaba algorithm.*

**JEL classification: C02, C44, C61.**

## 1. Introduction

The project management is an important and widely debated subject in the literature (see, e.g.: Kaufmann and Desbazeille, 1969; Liu, 2003; Doskočil and Doubravský, 2017), both theoretically and applicative. In this work we discuss about employing of mathematical modelling, particularly the critical path method (Kaufmann and Desbazeille, 1969) in this field. Generally, the structure of a project can be illustrated with a graph showing the durations of the activities and the dependencies between them. The critical path method consists in finding the minimum length of time in which all the activities are completed, that is the longest path between the first and the last stage. There are several ways to determine the critical path in a graph, one of which is the Bellman-Kalaba algorithm (Kaufmann and Desbazeille, 1969).

The critical path method can be used in a framework related to fuzzy sets. With regard to this approach, there are many recent papers (e.g.: Chanas and Ziełiński, 2001; Liu, 2003; Chen and Hsueh, 2008; Yakhchali, 2012). If the durations of the activities are fuzzy, some problems have to be overcome. It may be necessary to perform certain calculations with fuzzy numbers. This topic is



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addressed, for example, in Goetschel and Voxman (1986). Moreover, in the process of obtaining the critical path, we have to find a method to compare certain fuzzy numbers. There are various procedures of ranking fuzzy numbers (Detyniecki and Yager, 2001; Tran and Duckstein, 2002; Shureshjani and Darehmiraki, 2013; Ghorabaee et al., 2016).

In this paper we tackle the problem of finding the critical path when the activity times, that is the weights of the graph edges, are fuzzy. More precisely, we study the situation in which the durations are modeled using partially linearized Gaussian fuzzy numbers (Iacob and Popescu, 2011). The $\alpha$-level set of these fuzzy numbers is a closed interval. This feature is used in certain operations, as fuzzy addition and scalar multiplication. It also appears in the calculation of some crisp values associated with a fuzzy number. On the other hand, the partially linearized Gaussian numbers does not have the asymptotic behaviour of the Gaussian ones. This is due to a linearization procedure that will be discussed. Moreover, there exists partially linearized Gaussian numbers whose membership functions have domains that contains only positive elements with nonzero membership degree. This fits with the fact that the possible values of each activity duration are positive. The valuation (Yager, 1981; Yager and Filev, 1999) is a certain scalar value associated with a fuzzy number. We use valuations in the process of ranking fuzzy numbers of the proposed type. Finally, to determine the critical path in a directed graph with fuzzy weights, we use the Bellman-Kalaba algorithm. The results are generalized by establishing a correspondence between a Gaussian fuzzy number and a parameterized family of partially linearized Gaussian numbers.

The work is organized as follows. Section 2 contains certain theoretical aspects including relevant properties of the linearized Gaussian fuzzy numbers and a procedure of ranking them. The issue of the critical path method in a fuzzy context is also addressed. Section 3 focuses on a numerical application that validates the theoretical results.

## 2. Theoretical setting

### 2.1. Properties of partially linearized Gaussian fuzzy numbers

A Gaussian fuzzy number $\tilde{g} = \mathrm{gfn}(m, \sigma_L, \sigma_R)$ is defined by the membership function (Hanss, 2005)

$$g : \mathrm{R} \to [0,1], \; g(x) = \begin{cases} e^{\frac{-(x-m)^2}{2\sigma_L^2}}, & x \in (-\infty, m] \\ e^{\frac{-(x-m)^2}{2\sigma_R^2}}, & x \in (m, \infty), \end{cases}$$

where $m, \sigma_L, \sigma_R \in \mathrm{R}$ and $\sigma_L, \sigma_R > 0$. There are certain fuzzy numbers related to it, such as quasi-Gaussian numbers (Hanss, 2005) and partially linearized Gaussian numbers (Iacob and Popescu, 2011). A quasi-Gaussian number $\tilde{q}$ has the membership function

$$q:\mathrm{R}\to[0,1],\ q(x)=\begin{cases} g(x),\ x\in(m-3\sigma_L,m+3\sigma_R) \\ 0,\ x\in(-\infty,m-3\sigma_L]\cup[m+3\sigma_R,\infty). \end{cases}$$

For introducing a partially linearized Gaussian number, we consider a real number $a\in(0,1)$ and the corresponding values

$$l=g\big|_{(-\infty,m]}^{-1}(a)=m-\sigma_L\sqrt{-2\ln a}\ ,$$
$$r=g\big|_{[m,\infty)}^{-1}(a)=m+\sigma_R\sqrt{-2\ln a}\ ,$$

where $g\big|_{(-\infty,m]}$, $g\big|_{[m,\infty)}$ are restrictions of the function $g$.

Using the equations of the tangent lines to the graph of $g$ at the points $(l,a)$ and $(r,a)$, we obtain the membership function $h:\mathrm{R}\to[0,1]$ that characterize a partially linearized Gaussian number $\tilde{h}$. The function $h$ has a Gaussian form on the interval $\mathrm{I}_2=[l,r]$ and a piecewise linear appearance on $\mathrm{I}_1\cup\mathrm{I}_3$, where $\mathrm{I}_1=(-\infty,l]$ and $\mathrm{I}_3=[r,\infty)$. More precisely, we have

$$h(x)=\begin{cases} \dfrac{a\sqrt{-2\ln a}}{\sigma_L}(x-m)+a(1-2\ln a),\ x\in\left[l-\dfrac{\sigma_L}{\sqrt{-2\ln a}},l\right] \\ g(x),\ x\in(l,r] \\ \dfrac{-a\sqrt{-2\ln a}}{\sigma_R}(x-m)+a(1-2\ln a),\ x\in\left(r,r+\dfrac{\sigma_R}{\sqrt{-2\ln a}}\right] \\ 0,\ x\in\left(-\infty,l-\dfrac{\sigma_L}{\sqrt{-2\ln a}}\right)\cup\left(r+\dfrac{\sigma_R}{\sqrt{-2\ln a}},\infty\right). \end{cases}$$

The function $h$ is continuous and $h(\mathrm{R})=[0,1]$. Unlike Gaussian numbers, the fuzzy number $\tilde{h}$ does not have an asymptotic behaviour.

Because $\tilde{h}$ is uniquely determined by four real values, we make the notation $\tilde{h}=\mathrm{lgfn}\,(m,\sigma_L,\sigma_R,a)$.

Using the $\alpha$-level set (Goetschel and Voxman, 1986; Ming et al., 1997; Yager and Filev, 1999; Carlsson and Fullér, 2001)

$$\hat{\mathrm{h}}_\alpha=\begin{cases} \{x\mid h(x)\geq\alpha\}\text{ if }\alpha\in(0,1] \\ \text{the closure of }\{x\mid h(x)>\alpha\}\text{ if }\alpha=0, \end{cases}$$

one can obtain another description of $\tilde{h}$, which is equivalent to that given by the membership function.

We have

$$\hat{\mathrm{h}}_\alpha=\left[\underline{h}(\alpha),\overline{h}(\alpha)\right],$$

where

$$\underline{h}(\alpha)=\begin{cases} m-\sigma_L\sqrt{-2\ln a}-\dfrac{\sigma_L}{\sqrt{-2\ln a}}+\dfrac{\sigma_L}{a\sqrt{-2\ln a}}\alpha, \alpha\in[0,a) \\ m-\sigma_L\sqrt{-2\ln \alpha}, \alpha\in[a,1], \end{cases}$$

$$\overline{h}(\alpha)=\begin{cases} m+\sigma_R\sqrt{-2\ln a}+\dfrac{\sigma_R}{\sqrt{-2\ln a}}-\dfrac{\sigma_R}{a\sqrt{-2\ln a}}\alpha, \alpha\in[0,a) \\ m+\sigma_R\sqrt{-2\ln \alpha}, \alpha\in[a,1]. \end{cases}$$

*Example* 1. Consider $\tilde{g}=\mathrm{gfn}(24,8,5)$ and $\tilde{h}=\mathrm{lgfn}(24,8,5,0.12)$, with membership functions $g$ and $h$ (see Fig. 1). In this case, $h$ preserves the Gaussian form on the closed interval $\mathrm{I}_2=[7.5259,34.2962]$.

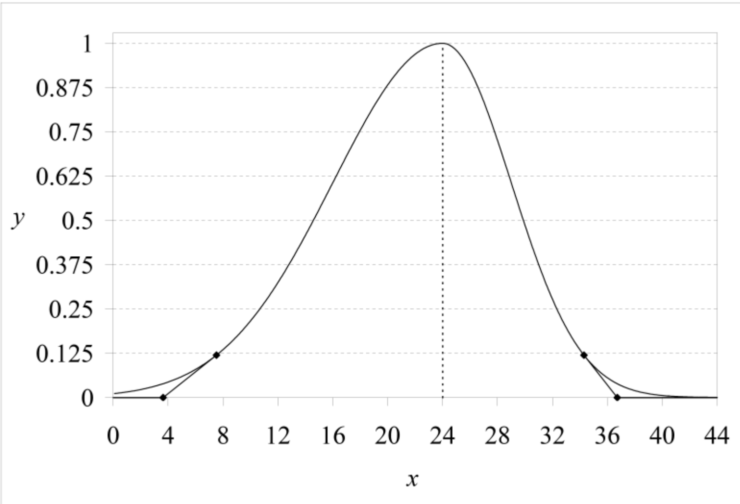


**Figure 1. The graphs of the functions $g$ and $h$.**

Let $\tilde{h}=\mathrm{lgfn}(m,\sigma_L,\sigma_R,a)$ and $\tilde{h}'=\mathrm{lgfn}(m',\sigma'_L,\sigma'_R,a)$ be two fuzzy numbers having the $\alpha$-level sets $[\underline{h}(\alpha),\overline{h}(\alpha)]$ and $[\underline{h}'(\alpha),\overline{h}'(\alpha)]$, respectively. We can add $\tilde{h}$ with $\tilde{h}'$ by using a general method (Goetschel and Voxman, 1986; Ming et al., 1997). The result of the addition is the fuzzy number $\tilde{h}''$, whose $\alpha$-level set $[\underline{h''}(\alpha),\overline{h''}(\alpha)]$ is determined by the relations

$$\underline{h''}(\alpha)=\underline{h}(\alpha)+\underline{h'}(\alpha),$$
$$\overline{h''}(\alpha)=\overline{h}(\alpha)+\overline{h'}(\alpha).$$

Finally, we have $\tilde{h}'' = \mathrm{lgfn}\left(m+m', \sigma_L+\sigma'_L, \sigma_R+\sigma'_R, a\right)$. The scalar multiplication can be described in a similar way (Iacob and Popescu, 2011).

We use the concept of valuation (Yager, 1981; Yager and Filev, 1999; Detyniecki and Yager, 2001) to ranking partially linearized Gaussian numbers. Generally, the valuation of a fuzzy number $\tilde{f}$ is the crisp value

$$\mathrm{Val}(\tilde{f}) = \int_0^1 \mathrm{Average}(\hat{\mathrm{f}}_\alpha) d\alpha ,$$

where

$$\mathrm{Average}(\hat{\mathrm{f}}_\alpha) = \frac{\underline{f}(\alpha) + \overline{f}(\alpha)}{2}$$

when $\alpha$ -level set $\hat{\mathrm{f}}_\alpha$ is closed interval.

The formula for Average $(\hat{\mathrm{f}}_\alpha)$ is different from the one written above when the set $\hat{\mathrm{f}}_\alpha$ is not a closed interval (Yager and Filev, 1999).

If we consider the number $\tilde{h} = \mathrm{lgfn}\left(m, \sigma_L, \sigma_R, a\right)$, we have $\hat{\mathrm{h}}_\alpha = \left[\underline{h}(\alpha), \overline{h}(\alpha)\right]$ and

$$\mathrm{Average}(\hat{\mathrm{h}}_\alpha) = \begin{cases} m - \dfrac{\sigma_L - \sigma_R}{2} \dfrac{1-2\ln a}{\sqrt{-2\ln a}} + \dfrac{\sigma_L - \sigma_R}{2} \dfrac{1}{a\sqrt{-2\ln a}} \alpha, \alpha \in [0, a] \\ m - \dfrac{\sigma_L - \sigma_R}{2} \sqrt{-2\ln \alpha}, \alpha \in (a, 1]. \end{cases}$$

Therefore, we obtain

$$\mathrm{Val}(\tilde{h}) = \int_0^a \left( m - \frac{\sigma_L - \sigma_R}{2} \frac{1-2\ln a}{\sqrt{-2\ln a}} + \frac{\sigma_L - \sigma_R}{2} \frac{1}{a\sqrt{-2\ln a}} \alpha \right) d\alpha +$$

$$+ \int_a^1 \left( m - \frac{\sigma_L - \sigma_R}{2} \sqrt{-2\ln \alpha} \right) d\alpha =$$

$$= m - \frac{\sigma_L - \sigma_R}{4} \frac{a(1-4\ln a)}{\sqrt{-2\ln a}} - \frac{\sigma_L - \sigma_R}{2} J(a),$$

where

$$J(a) = \int_a^1 \sqrt{-2\ln \alpha}\, d\alpha .$$

Two properties of the lower incomplete gamma function

$$\gamma(p,y)=\int_0^y t^{p-1}e^{-t}dt,\ p>0,\ y>0$$

can be stated as follows (Abramowitz and Stegun, 1972):

$$\gamma(p+1,y)=-y^p e^{-y}+p\gamma(p,y),$$
$$\gamma\left(\frac{1}{2},y\right)=\sqrt{\pi}\,\mathrm{erf}\left(\sqrt{y}\right),$$

where

$$\mathrm{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt$$

is the error function.

The substitution $\alpha=e^{-t}$ is considered in order to compute the integral $J(a)$. Thus, one can obtain

$$J(a)=\sqrt{2}\int_0^{-\ln a} t^{1/2}e^{-t}dt=\sqrt{2}\gamma\left(\frac{3}{2},-\ln a\right)=$$
$$=-a\sqrt{-2\ln a}+\frac{1}{\sqrt{2}}\gamma\left(\frac{1}{2},-\ln a\right)=$$
$$=-a\sqrt{-2\ln a}+\sqrt{\frac{\pi}{2}}\mathrm{erf}\left(\sqrt{-\ln a}\right).$$

One of the possible approximations of the error function is given by the formula

$$\mathrm{erf}(z)\approx 1-\frac{1}{\left(1+c_1z+c_2z^2+\ldots+c_6z^6\right)^{16}},\ z\geq 0,$$

where the coefficients $c_k$, $k=\overline{1,6}$, are specified (Abramowitz and Stegun, 1972). The error of this approximation does not exceed the threshold $3\times 10^{-7}$. There is also computer software that gives the values of $\mathrm{erf}(z)$.

*Example* 2. In the case $\tilde{h}=\mathrm{lgfn}(24,8,5,0.12)$ (see also example 1), we can make the calculations: $\mathrm{erf}\left(\sqrt{-\ln 0.12}\right)=0.9605$, $J(0.12)=0.9567$, $\mathrm{Val}\left(\tilde{h}\right)=22.1505$.

A partially linearized Gaussian number $\tilde{h}=\mathrm{lgfn}(m,\sigma_L,\sigma_R,a)$ has the property that all real values with nonzero membership degree are positive if and only if the following constraint related to the left spread holds:

$$\xi_L(\tilde{h})\geq 0,$$

where

$$\xi_L(\tilde{h}) = m - \sigma_L\sqrt{-2\ln a} - \frac{\sigma_L}{\sqrt{-2\ln a}}.$$

Remark that $\mathrm{lgfn}(24,8,5,0.12)$ verifies the condition, because $3.641 > 0$.

**2.2. Parameterized fuzzy numbers**

In a more general framework, different values of the real number $a$ can be considered according to different circumstances. Therefore $a$ can be viewed as a parameter belonging to a certain set $\mathrm{A} \subseteq (0,1)$. In this way an arbitrary and fixed Gaussian fuzzy number $\tilde{g} = \mathrm{gfn}(m, \sigma_l, \sigma_r)$ can be associated with the set

$$\mathrm{H}_a = \left\{ \tilde{h}(a) \,\middle|\, \tilde{h}(a) = \mathrm{lgfn}(m, \sigma_L, \sigma_R, a) \text{ and } a \in \mathrm{A} \right\}.$$

This allows a more elastic modelling of various fuzzy data by adjusting the value of the parameter $a$. Under the assumption that $\mathrm{A}$ has the largest possible range, namely $\mathrm{A} = (0,1)$, we obtain

$$\frac{\partial \mathrm{Val}(\tilde{h}(a))}{\partial a} = \frac{\sigma_L - \sigma_R}{8} \frac{1 + 2\ln a}{\sqrt{-2\ln a}\ln a},$$

$$\frac{\partial^2 \mathrm{Val}(\tilde{h}(a))}{\partial a^2} = -\frac{\sigma_L - \sigma_R}{16} \frac{3 + 2\ln a}{a\sqrt{-2\ln a}(\ln a)^2}.$$

In the symmetrical case, that is $\sigma_L = \sigma_R$, we have $\mathrm{Val}(\tilde{h}(a)) = m$, $\forall a \in (0,1)$.

If $\sigma_L < \sigma_R$, then $\mathrm{Val}(\tilde{h}(a))$ is decreasing when $a \in \left(0, e^{-1/2}\right]$ and is increasing when $a \in \left[e^{-1/2}, 1\right)$. Therefore it results that

$$\min_{a \in (0,1)} \mathrm{Val}(\tilde{h}(a)) = \mathrm{Val}\left(\tilde{h}\left(e^{-1/2}\right)\right).$$

If $\sigma_L > \sigma_R$, then $\mathrm{Val}(\tilde{h}(a))$ is increasing on $\left(0, e^{-1/2}\right]$ and decreasing on $\left[e^{-1/2}, 1\right)$. In this situation we deduce that

$$\max_{a \in (0,1)} \mathrm{Val}(\tilde{h}(a)) = \mathrm{Val}\left(\tilde{h}\left(e^{-1/2}\right)\right).$$

Moreover, we have

$$\mathrm{Val}\left(\tilde{h}\left(e^{-1/2}\right)\right) = m - \frac{\sigma_L - \sigma_R}{4}\left(e^{-1/2} + \sqrt{2\pi} \cdot \mathrm{erf}\left(\frac{\sqrt{2}}{2}\right)\right).$$

Further, we give a procedure for study the existence of a subset $\mathrm{H}'_a$ of $\mathrm{H}_a$, consisting of partially linearized Gaussian numbers that verifies the constraint

$$\xi_L(\tilde{h}(a)) \geq 0.$$

This inequality is equivalent to

$$-\sigma_L\left(\sqrt{-2\ln a}\right)^2 + m\sqrt{-2\ln a} - \sigma_L \geq 0.$$

At first, remark that $2\sigma_L \le m$ is a necessary condition for $\mathrm{H}'_a$ to be nonempty. From here we have two cases, as follows.

*Case* 1. If $2\sigma_L = m$, then $\tilde{h}\left(e^{-1/2}\right)$ is the unique partially linearized Gaussian number that satisfy the constraint.

*Case* 2. If $2\sigma_L < m$, the set of the feasible values of the parameter $a$ is $\mathrm{A} \cap \mathrm{A}'$, where $\mathrm{A}' = \left(e^{-\lambda_L}, e^{-\mu_L}\right)$ and

$$\lambda_L = \frac{1}{4\sigma_L^2}\left(m^2 + m\sqrt{m^2 - 4\sigma_L^2} - 2\sigma_L^2\right),$$

$$\mu_L = \frac{1}{4\sigma_L^2}\left(m^2 - m\sqrt{m^2 - 4\sigma_L^2} - 2\sigma_L^2\right).$$

When $\mathrm{A} \cap \mathrm{A}'$ is nonempty, $\mathrm{H}'_a$ is generated by the elements $a \in \mathrm{A} \cap \mathrm{A}'$.

### 2.3. Critical path in a fuzzy setting

Consider a project with the network described by a directed graph without loops or circuits $\mathrm{G} = (\mathrm{V}, \mathrm{E})$, where $\mathrm{V} = \{v_1, ..., v_n\}$ is the set of vertices and $\mathrm{E}$ is the set of edges. Every directed edge $\left(v_i, v_j\right)$ has the associated weight $\tilde{h}_{i,j}(a)$ that is a partially linearized Gaussian number representing the time to reach the stage $x_j$, starting from the level $x_i$. To ensure the nonnegativity of all possible values of the fuzzy duration $\tilde{h}_{i,j}(a)$, it is necessary to take into account the restriction

$$\xi_L(\tilde{h}_{i,j}(a)) \ge 0.$$

We propose the use of the Bellman-Kalaba algorithm (Kaufmann and Desbazeille, 1969) to find the critical path. The fuzzy numbers representing the weights of the edges or of the various paths can be ranked using their valuations. Moreover, taking into account the properties of the linearized Gaussian fuzzy numbers and the definition of the valuation, we have the relation

$$\mathrm{Val}\left(\tilde{h}_{i_1,j_1}(a) + \tilde{h}_{i_2,j_2}(a)\right) = \mathrm{Val}\left(\tilde{h}_{i_1,j_1}(a)\right) + \mathrm{Val}\left(\tilde{h}_{i_2,j_2}(a)\right),$$

where $i_1, i_2, j_1, j_2 \in \{1,,.n\}$ and $\left(v_{i_1}, v_{j_1}\right), \left(v_{i_2}, v_{j_2}\right) \in \mathrm{E}$.

### 3. Numerical application

Let $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ be a graph having the structure (Fig. 2) similar to that used by Chanas and Ziełiński (2001). This type of project network was also employed in other papers (Chen and Hsueh, 2008; Yakhchali, 2012).

The durations of the activities, i.e. the edge weights, are considered to be partially linearized Gaussian fuzzy numbers depending on the parameter $a$, subject to certain constraints, as it is shown in the previous section. We assume the following fuzzy weights:

$\tilde{h}_{1,2}(a) = \text{lgfn}(36,6,2,a)$, $\tilde{h}_{1,3}(a) = \text{lgfn}(30,3,7,a)$, $\tilde{h}_{2,4}(a) = \text{lgfn}(32,10,3,a)$,
$\tilde{h}_{2,5}(a) = \text{lgfn}(43,5,11,a)$, $\tilde{h}_{3,4}(a) = \text{lgfn}(21,3,11,a)$, $\tilde{h}_{3,6}(a) = \text{lgfn}(46,12,6,a)$,
$\tilde{h}_{4,6}(a) = \text{lgfn}(37,1,8,a)$, $\tilde{h}_{4,7}(a) = \text{lgfn}(40,4,9,a)$, $\tilde{h}_{5,9}(a) = \text{lgfn}(8,2,4,a)$,
$\tilde{h}_{5,11}(a) = \text{lgfn}(24,8,5,a)$, $\tilde{h}_{6,8}(a) = \text{lgfn}(48,8,3,a)$, $\tilde{h}_{7,8}(a) = \text{lgfn}(44,12,4,a)$,
$\tilde{h}_{8,10}(a) = \text{lgfn}(12,3,2,a)$, $\tilde{h}_{8,11}(a) = \text{lgfn}(23,4,2,a)$, $\tilde{h}_{9,11}(a) = \text{lgfn}(15,5,8,a)$,
$\tilde{h}_{10,11}(a) = \text{lgfn}(19,2,3,a)$.

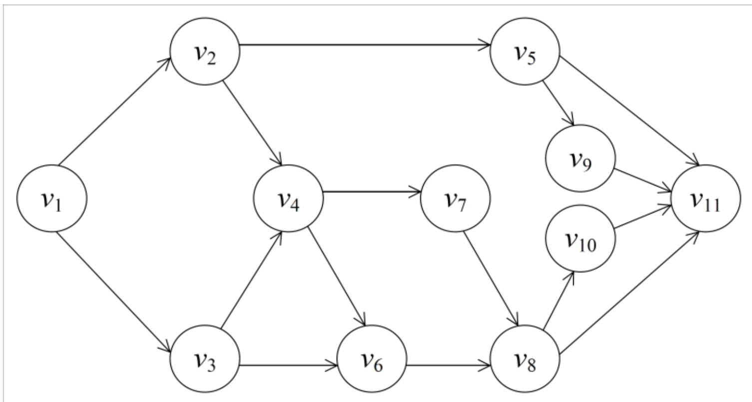


**Figure 2. The project network**.

At first, we analyze the case $a = 0.1$. The condition $\xi_L(\tilde{h}_{i,j}(0.1)) \geq 0$ is satisfied for each weight that corresponds to an edge in $\mathrm{G} = (\mathrm{V}, \mathrm{E})$. After some calculations, we obtain the following results:
$\mathrm{Val}(\tilde{h}_{1,2}(0.1)) = 33.52667$, $\mathrm{Val}(\tilde{h}_{1,3}(0.1)) = 32.47332$, $\mathrm{Val}(\tilde{h}_{2,4}(0.1)) = 27.67167$,
$\mathrm{Val}(\tilde{h}_{2,5}(0.1)) = 46.70998$, $\mathrm{Val}(\tilde{h}_{3,4}(0.1)) = 25.94665$, $\mathrm{Val}(\tilde{h}_{3,6}(0.1)) = 42.29001$,
$\mathrm{Val}(\tilde{h}_{4,6}(0.1)) = 41.32832$, $\mathrm{Val}(\tilde{h}_{4,7}(0.1)) = 43.09165$, $\mathrm{Val}(\tilde{h}_{5,9}(0.1)) = 9.23666$,
$\mathrm{Val}(\tilde{h}_{5,11}(0.1)) = 22.14500$, $\mathrm{Val}(\tilde{h}_{6,8}(0.1)) = 44.90834$, $\mathrm{Val}(\tilde{h}_{7,8}(0.1)) = 39.05334$,
$\mathrm{Val}(\tilde{h}_{8,10}(0.1)) = 11.38166$, $\mathrm{Val}(\tilde{h}_{8,11}(0.1)) = 21.76333$, $\mathrm{Val}(\tilde{h}_{9,11}(0.1)) = 16.85499$,
$\mathrm{Val}(\tilde{h}_{10,11}(0.1)) = 19.61833$.
Using the Bellman-Kalaba algorithm, we find that the critical path in the graph $\mathrm{G} = (\mathrm{V}, \mathrm{E})$ goes through the vertices $v_1$, $v_2$, $v_4$, $v_6$, $v_8$, $v_{10}$, $v_{11}$, in this order. In addition, the total fuzzy duration of the critical path $\tilde{h}_{\mathrm{cp}}$ can be computed as the

sum of the fuzzy weights of its component edges. Therefore, in this case, $\tilde{h}_{\text{cp}}(0.1) = \text{lgfn}(184,30,21,0.1)$ having the valuation $\text{Val}(\tilde{h}_{\text{cp}}(0.1)) = 178.43501$ .

If we consider the discrete set $\mathrm{A} = \{0.05k \mid k = \overline{1,18}\}$ then, for every fuzzy number $\tilde{h}_{i,j}(a)$, we have the relation $\mathrm{A} \cap \mathrm{A}'_{i,j} = \mathrm{A}$ . For all $a \in \mathrm{A}$ , the critical path has the vertices $v_1$ , $v_2$ , $v_4$ , $v_6$ , $v_8$ , $v_{10}$ , $v_{11}$. Its weight is the fuzzy number $\tilde{h}_{\text{cp}}(a)$, which can be obtained in the same way as $\tilde{h}_{\text{cp}}(0.1)$. Remark that the critical path remains the same but its weight changes when $a$ takes different values. We have $\text{Val}(\tilde{h}_{\text{cp}}(0.05)) = 178.39520$ , $\text{Val}(\tilde{h}_{\text{cp}}(0.6)) = 178.78484$ , $\text{Val}(\tilde{h}_{\text{cp}}(0.9)) = 177.59326$ . The complete image of the variation of the valuation of $\tilde{h}_{\text{cp}}(a)$ depending on the values of the parameter $a \in \mathrm{A}$ is given in Fig. 3. One can remark that the calculations in this particular case are consistent with general theoretical results obtained when $\mathrm{A} = (0,1)$.

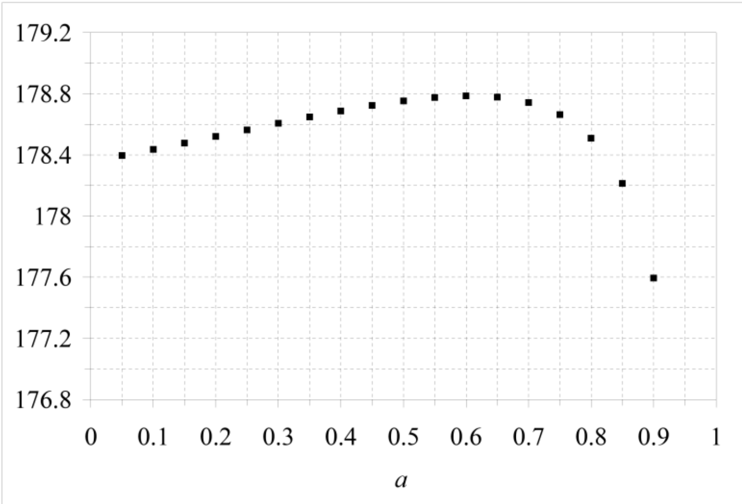


**Figure 3. Critical path valuation as a function of $a$ .**

Further, we study the situation in which the model discussed above has four weights modified as follows: $\tilde{h}_{1,2}(a) = \text{lgfn}(36,4,10,a)$, $\tilde{h}_{1,3}(a) = \text{lgfn}(43,9,3,a)$, $\tilde{h}_{2,4}(a) = \text{lgfn}(64,3,13,a)$ and $\tilde{h}_{3,4}(a) = \text{lgfn}(78,18,4,a)$. Hence we obtain the results: $\text{Val}(\tilde{h}_{1,2}(0.1)) = 39.70998$, $\text{Val}(\tilde{h}_{1,3}(0.1)) = 39.29001$, $\text{Val}(\tilde{h}_{2,4}(0.1)) = 70.18331$ and $\text{Val}(\tilde{h}_{3,4}(0.1)) = 69.34335$ . In this case, the critical path vertices doesn't remain the same for all $a \in \mathrm{A}$. Consider the sets $\mathrm{B} = \left\{0.05k \,\middle|\, k = \overline{10,13}\right\}$ and $\mathrm{C} = \mathrm{A} - \mathrm{B}$. The critical path goes through $v_1$, $v_3$, $v_4$, $v_6$, $v_8$, $v_{10}$, $v_{11}$ when $a \in \mathrm{B}$ and it goes through $v_1$, $v_2$, $v_4$, $v_6$, $v_8$, $v_{10}$, $v_{11}$ when $a \in \mathrm{C}$. For example, if we consider the values $0.1 \in \mathrm{C}$ and $0.55 \in \mathrm{B}$, we obtain $\tilde{h}_{\mathrm{cp}}(0.1) = \mathrm{lgfn}(216,21,39,0.1)$ and $\tilde{h}_{\mathrm{cp}}(0.55) = \mathrm{lgfn}(237,41,23,0.55)$. Moreover, we have $\mathrm{Val}(\tilde{h}_{\mathrm{cp}}(0.1)) = 227.12996$ and $\mathrm{Val}(\tilde{h}_{\mathrm{cp}}(0.55)) = 226.54952$. The different valuations of the critical path obtained when $a \in \mathrm{A}$ are shown in Fig. 4.

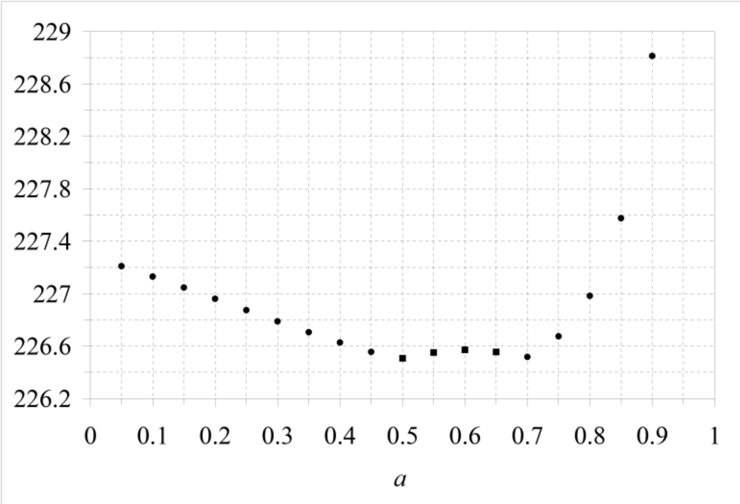


**Figure 4. Critical paths valuations for $a \in \mathrm{B} \cup \mathrm{C}$.**

### 4. Conclusions

The partially linearized Gaussian fuzzy numbers were presented and some of their properties have been analyzed. A parameterization method and a ranking procedure were considered. Also, a numerical application consisting in finding the critical path in a network with fuzzy durations was given.

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